Theory of Estimation

$\mathcal{E}$: Random Experiment e.g. tossing of a coin.

X : Random observation or outcome of the experiment, may be vector-valued.

$\mathfrak{X}$: $(x_1, x_2, \cdots, x_n)$, The sample space.

e.g. For n independent Bernoullian trials with common unknown probability of success θ,

$\mathfrak{X} = \{(x_1, x_2, \cdots, x_n) ; x_i = 0 \text{ or } 1, i = 1(1)n\}$, where 0 denotes failure and 1 denotes success. $p(x) \in \{\theta^{\sum x_i}(1-\theta)^{n-\sum x_i} ; \theta \in \Omega\}$; $\Omega = (0,1)$. In this case, θ is a real-valued parameter.

e.g. $\mathfrak{X} = \{(x_1, x_2, \cdots, x_n)\}$; x_i's, $i=1(1)n$ are independent observations from $N(\mu, \sigma^2)$ with $\underset{\sim}{\theta} = (\mu, \sigma^2)$ unknown. In this case $\underset{\sim}{\theta}$ is vector-valued parameter.

$\mathfrak{X} = \mathbb{R}^n$, n-dimentional real space.

$F(x)$: $P(X \le x)$, completely known except for some parameter θ. This means that $F(x)$ belongs to the family $\{F_\theta(x) ; \theta \in \Omega\}$, a family of parametric distributions.

Ω : Set of all possible values of θ.

Example: $X \sim N(\theta, 1)$; $-\infty < \theta < \infty$. So $\Omega = (-\infty, \infty)$.

Let $\underset{\sim}{X} = (x_1, x_2, \cdots, x_n)$ be n independent observations from a population which is characterized by an unknown univariate pdf $f(x)$.

Here $\mathfrak{X} = \mathbb{R}^n$, n-dimentional real space.

$p(\mathbf{x}) \in \{\prod_{i=1}^{n} f(x_i) : f \text{ is any univariate pdf}\}$.

$\theta = f$ is an abstract valued parameter.

Ω = class of all possible univariate pdfs.

Problems of Estimation

1. Point Estimation: Here we have selected one value of θ i.e. one particular member of the family of distributions or pmf (pdf) which seems most appropriate in view of the observations $\underset{\sim}{x}$,

$p(x) \in \{p_\theta(x), \theta \in \Omega\}$, θ is unknown.

If $T(x)$ stands for this choice of θ, then $T(x)$ should be as close to the true value of θ as possible.

This is the problem of point estimation.

2. Set Estimation or Interval Estimation:

Here we have to select, on the basis of $\underset{\sim}{x}$, a subset of Ω {i.e. a subset of the family of distributions or pmf (or pdf)}, say $S(x)$ such that we can say with certain confidence the true value of θ lies in $S(x)$. $S(x)$ should be as short as possible in some sense.

Example: $X \sim N(\theta, 1)$, $\Omega = (-\infty, \infty)$

$P[\hat{\theta}_1 < \theta < \hat{\theta}_2] = 1-\alpha$

Confidence interval $(\hat{\theta}_1, \hat{\theta}_2) = S(x)$

This is the problem of set estimation or interval estimation.

The problem of estimation is called parametric problem if we are to estimate θ or, more generally, a function of θ, say $g(\theta)$, when θ is real valued or vector valued.

The problem is called nonparametric problem if we are to estimate a real or vector valued function of θ, say $g(\theta)$, when θ is abstract valued. e.g. estimation of $\mu(f)$ or $(\mu(f), \sigma^2(f))$, where

$$\mu(f) = \int_{-\infty}^{\infty} x f(x)dx, \quad \sigma^2(f) = \int_{-\infty}^{\infty} (x-\mu(f))^2 f(x)dx$$

§: Statistic: If $t(x)$ be a single valued function of x defined on $\mathcal{X}$, then $T(x)$ is called a Statistic.

A statistic $T(x)$ may be real valued or vector valued. The dimention of T is the number of coordinates in T.

The statistic T is used to reduce the original observation x.

Example: $\underset{\sim}{x} = (x_1, x_2, \cdots, x_n)$

$T_1 = x = (x_1, x_2, \cdots, x_n) \to$ n-dimentional statistic.

$T_2 = (x_{(1)}, x_{(2)}, \cdots, x_{(n)}) \to$ n-dimentional statistics, where $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$.

$=$ Order statistic

$T_3 = (\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \ \Sigma(x_i - \bar{x})^2) \to$ 2-dimentional statistic

$T_4 = \bar{x} \to$ 1-dimentional statistic.

$T_5 = \sum_{i=1}^{n} x_i \to$ one dimentional statistic.

Let T and T^* be two statistics such that $T^*(x)$ is a function of $T(x)$. Then we say that T^* gives a more thorough reduction of original data than T, and clearly T^* can be computed from the knowledge of T and not conversely.

Example: T_2 is a function of T_1.
T_5 is a function of both T_1 and T_2.

Equivalent Statistics: — T and T^* are said to ~~have~~ be equivalent statistics if they are of one to one relationship. In this case T is as useful as T^* and one can be computed from the knowledge of the other. e.g. T_4 and T_5 are equivalent statistic.

Sufficient statistic: Suppose we have a random variable (or vector) $\underset{\sim}{X}$ with pmf or pdf $p(x) \in \mathcal{P} = \{p_\theta(x) : \theta \in \Omega\}$, θ is unknown and we want ~~it~~ to infer about it on the basis of x.

x is generally bulk in nature. So a statistic $T = t(x)$ is used to reduce x to some convenient form. Here T should be so chosen as not to loose any information contained in x. Such a statistic is called a sufficient statistic.

Example: Let $x = (x_1, x_2, \cdots, x_n)$ be results of n Bernoulli trials $x_i = 0, 1$.

$$p_\theta(x) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}, \quad \theta = P(x_i = 1),\ \forall i = 1(1)n.$$

Consider $T = \Sigma x_i \sim Bin(n, \theta)$.

$$p_\theta^T(t) = \binom{n}{t}\theta^t(1-\theta)^{n-t}; \quad t = 0, 1, 2, \cdots, n.$$

$$P_\theta[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n / T = t]$$

$$= \frac{P_\theta[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n, T = t]}{P_\theta[T = t]}$$

$$= \begin{cases} \theta^t(1-\theta)^{n-t} \Big/ \binom{n}{t}\theta^t(1-\theta)^{n-t} & ; \text{ if } \sum_{i=1}^{n} x_i = t \\ 0 & ; \text{ otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\binom{n}{t}} & , \text{ if } \Sigma x_i = t \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ Beyond T, x does not add any further information about θ

$\Rightarrow$ T is a sufficient statistic for θ.

Formal defination of Sufficient Statistics

Defination 1: A statistic T (which may be vector valued) is said to be sufficient for $\mathcal{P}$ (or simply for θ) if the conditional distribution of X given $T=t$ is independent of θ for every admissible value t of T.

Defination 2: T is said to be sufficient for θ if the conditional distribution of any other statistic T_1 given $T=t$ is independent of θ for all admissable value t of T.

Defination 1 and Defination 2 are equivalent.

Proof: To show $(2) \Rightarrow (1)$

In def^n (2) take $T_1 = X \Rightarrow$ def^n (1)

To show $(1) \Rightarrow (2)$

Def^n (1) $\Rightarrow P_\theta[X \in A / T=t]$ is independent of θ $\forall t, \forall A \subset \mathcal{X}$

Take T_1, to be any other statistic.

Let $\mathcal{X}_1$ = Sample space of T_1 and consider any $B \subset \mathcal{X}_1$.

Then, $P_\theta[T_1 \in B / T=t] = P_\theta[X \in T_1^{-1}(B) / T=t]$, where $T_1^{-1}(B) \subset \mathcal{X}$

and this is independent of θ by def^n (1).

$\Rightarrow$ Def^n (2)

Thus, def^n (1) $\Leftrightarrow$ def^n (2).

Notes:

1. T is sufficient for $\mathcal{P} = \{p_\theta(x) : \theta \in \Omega\}$
   $\Rightarrow T$ is sufficient for $\mathcal{P}^* = \{p_\theta(x) : \theta \in \Omega^*\}$, $\Omega^* \subset \Omega$
   $\subset \mathcal{P}$.
2. If T and T^* are equivalent statistics, ~~then~~ and T is sufficient for θ, then T^* is sufficient for θ.

~~Proof: Since T and T* are equivalent statistics~~

3. Let T and T^* be two statistics such that T is a function of T^*. Then T is sufficient for θ implies T^* is sufficient for θ.
4. X is always a sufficient statistic. //

Ex1. ~~Examples~~ $X \sim P(\theta)$

$$p_\theta(\underline{x}) = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}, \quad x_i = 0,1,2,\cdots$$

Let $T = \sum_{i=1}^{n} x_i \sim P(n\theta)$

$$p_\theta^T(t) = \frac{e^{-n\theta}(n\theta)^t}{t!}; \quad t = 0,1,2,\cdots$$

$$P_\theta[X_1=x_1, X_2=x_2, \cdots, X_n=x_n / T=t]$$

$$= \frac{P_\theta[X_1=x_1, X_2=x_2, \cdots, X_n=x_n, T=t]}{p_\theta^T(t)} = \frac{e^{-n\theta}\,\theta^t}{\prod_{i=1}^{n} x_i!} \times \frac{t!}{e^{-n\theta}(n\theta)^t} = \begin{cases} \dfrac{t!}{n^t \prod_{i=1}^{n} x_i!} & \text{if } \sum x_i = t \\ 0 & \text{if } \sum x_i \neq t \end{cases}$$

which is independent of θ.

$\Rightarrow T = \sum x_i$ is sufficient for θ.

Ex 2. Suppose we have N items, θ of which is defective. θ is unknown. Let n items be drawn by SRSWOR. Let us define

$$x_i = 1 \text{ if } i\text{th selected item is defective.}$$
$$= 0 \text{ if } i\text{th selected item is not defective.}$$

Let us take $T = \sum_{i=1}^{n} x_i$ = No. of defective items in the sample.

For given, $\sum_{i=1}^{n} x_i$,

$$P_\theta[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n] = P_\theta[X_1 = x_{i_1}, X_2 = x_{i_2}, \cdots, X_n = x_{i_n}],$$

where $(i_1, i_2, \cdots, i_n)$ be any permutation of $(1, 2, \cdots, n)$.

$$P_\theta[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n / T = t]$$
$$= \frac{P_\theta[X_1 = x_1, \cdots, X_n = x_n, T = t]}{P_\theta[T = t]}$$
$$= \frac{P_\theta[X_1 = 1, X_2 = 1, \cdots, X_{t(x)} = 1, X_{t(x)+1} = 0, \cdots, X_n = 0]}{P_\theta[T = t]}$$
$$= \left[\frac{\theta}{N} \times \frac{\theta-1}{N-1} \times \frac{\theta-2}{N-2} \times \cdots \times \frac{\theta-t+1}{N-t+1} \times \frac{N-\theta}{N-t} \times \frac{N-\theta-1}{N-t-1} \times \cdots \times \frac{N-\theta-n+t+1}{N-n+1}\right] \Big/ \frac{\binom{\theta}{t}\binom{N-\theta}{n-t}}{\binom{N}{n}}$$

$[\because T \sim Hyp.G(N, n; \theta)]$

$$\text{Denominator} = \binom{\theta}{t}\binom{N-\theta}{n-t} \Big/ \binom{N}{n}$$
$$= \frac{\theta(\theta-1)\cdots(\theta-t+1)}{t!} \times \frac{(N-\theta)(N-\theta-1)\cdots(N-\theta-n+t+1)}{(n-t)!} \times \left[\frac{N(N-1)\cdots(N-n+1)}{n!}\right]^{-1}$$

Conditional probability

$$= \frac{N!}{\binom{n}{t}(N-n)!} \times \frac{1}{(N)_n} = \begin{cases} \frac{1}{\binom{n}{t}} & \text{if } \sum x_i = t \\ 0 & \text{, otherwise.} \end{cases}$$

It is independent of θ, so $T = \sum_{i=1}^{n} x_i$ is sufficient statistic for θ.

The method of finding a sufficient statistic by computing the conditional distribution is a very labourious method. A simpler method has been proposed by Neyman and it goes by the name of Neyman's Factorization Theorem.

Neyman's Factorization Criterion

Neyman's Factorization ~~Theorem~~ Criterion

Theorem: A statistic T is said to be sufficient for $\mathcal{P} = \{p_\theta(x) : \theta \in \Omega\}$ iff we can write

$$p_\theta(x) = g_\theta(t) \cdot h(x) \quad \forall \theta \quad ----(1)$$

where the first term may depend on θ but depends on x only through T and the second term is independent of θ.

Examples:

1. $p_\theta(\underline{x}) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$; $x_i = 0, 1$, $0 < \theta < 1$.

$$= g_\theta(\Sigma x_i) \cdot h(x), \text{ where } g_\theta(\Sigma x_i) = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$$
$$h(\underline{x}) = 1$$

$\Rightarrow T = \Sigma x_i$ is a sufficient statistic.

2. $p_\theta(\underline{x}) = e^{-n\theta}\, \theta^{\Sigma x_i} / \prod_{i=1}^{n} x_i!$

$$= g_\theta(\Sigma x_i) \cdot h(\underline{x}), \text{ where } g_\theta(\Sigma x_i) = e^{-n\theta}\,\theta^{\Sigma x_i} \text{ and } h(\underline{x}) = \frac{1}{\prod x_i!}$$

$\Rightarrow T = \Sigma x_i$ is a sufficient statistic.

Corollary 1. If T and T^* be such that T is a function of T^*, then T is sufficient for $\theta \Rightarrow T^*$ is sufficient for θ.

Proof: Let $T = \psi(T^*)$

T is sufficient for θ.

$$\Rightarrow p_\theta(\underline{x}) = g_\theta(t(x)) \cdot h(x)$$
$$= g_\theta(\psi(t^*(x))) \cdot h(x)$$
$$= g_{\theta^*}(t^*(x)) \cdot h(x), \text{ where } g_\theta(\psi(\cdot)) = g_{\theta^*}(\cdot)$$

$\Rightarrow T^*$ is a sufficient statistic for θ.

Corollary 2. If T and T^* be equivalent statistics, then T is sufficient for $\theta \Leftrightarrow T^*$ is sufficient for θ.

Proof of the Factorization theorem:

1. Discrete Case:

If part

Suppose (1) holds

Then,

Then, $p_\theta^T(t) = \sum_{x': t(x')=t} p_\theta(x') = g_\theta(t) \sum_{x': t(x')=t} h(x')$

Hence, $$P_\theta[X=x/T=t] = \frac{P_\theta[X=x, T=t]}{P_\theta[T=t]}$$

$$= \begin{cases} \dfrac{h(x)}{\sum_{x': t(x')=t} h(x')} & \text{if } t(x)=t \\ 0 & \text{if } t(x)\neq t \end{cases}$$

which is independent of θ.

$\Rightarrow$ T is sufficient for θ.

Only if part

Let $P_\theta[X=x/T=t]$ is independent of θ, say, equal $K(x,t)$.

Then, $p_\theta(x) = p_\theta^T(t) . P_\theta[X=x/T=t]$

$= p_\theta^T(t) . K(x,t)$

$= g_\theta(t) . h(x)$, where $g_\theta(t) = p_\theta^T(t)$, $h(x) = K(x, t(x))$.

II. Absolutely continuous case:

Let $X = (x_1, x_2, \ldots, x_n)$, $T = (T_1, T_2, \ldots, T_r)$, $r<n$.

Let there exist $Y = (Y_1, Y_2, \ldots, Y_{n-r}) \ni$ transformation $X \to (T, Y)$ is 1:1.

Then $p_\theta(x) = g_\theta(t_1(x), t_2(x), \ldots, t_r(x), y_1(x), y_2(x), \ldots, y_{n-r}(x))$.
$$J\left(\frac{t_1, t_2, \ldots, t_r, y_1, y_2, \ldots, y_{n-r}}{x_1, x_2, \ldots, x_n}\right)$$

Assuming $J(\neq 0)$ exists.

Then $p_\theta^{Y/t}(y)$ = conditional distribution of Y given $T=t$.

$= p_\theta^{T,Y}(t,y) \,/\, p_\theta^T(t)$

$= p_\theta^{T,Y}(t,y) \big/ \int p_\theta^{T,Y}(t,y')\,dy'$ --- (2)

Now T is sufficient for θ

$\Rightarrow$ The conditional distribution of Y given $T=t$ is independent of θ i.e. (2) is independent of θ

Conversely,

(2) is independent of θ i.e. The conditional distn. of Y given $T=t$ is indp of θ.

$\Rightarrow P_\theta[Y \in B/T=t]$ is indept. of θ $\forall B \subset \mathcal{X}^{Y/t}$ ---- (3)

$P_\theta\{X \in A/T=t\} = P_\theta\{(T,Y) \in C/T=t\}$, where $C = \{(t,y) \mid x \in A\}$

$= P_\theta\{Y \in B/T=t\}$, where $B = \{y/(t,y) \in C\}$

(3) $\Rightarrow P_\theta\{X \in A/T=t\}$ is independent of θ $\quad \forall A \in \mathcal{X}$

$\Rightarrow$ T is sufficient for θ.

Hence to prove the theorem, it is sufficient to show that (2) is independent of θ iff (1) holds.

<u>If part:</u>

Let (1) holds

Then, $p_\theta^{T,Y}(t,y) = p_\theta\left(x_1(t,y), x_2(t,y), \ldots, x_n(t,y)\right) \times J\left(\frac{x_1, x_2, \ldots, x_n}{t_1, \ldots t_r, y_1, \ldots y_{n-r}}\right)$

$= g_\theta(t_1, t_2, \ldots, t_r)\, h\left(x_1(t,y), x_2(t,y), \ldots, x_n(t,y)\right) . J\left(\frac{x_1, x_2, \ldots, x_n}{t_1, \ldots t_r, y_1, \ldots y_{n-r}}\right)$

$= g_\theta(t) . K(t,y)$, say, where $K(t,y)$ is independent of θ.

Then, (2) $= \dfrac{p_\theta^{T,Y}(t,y)}{p_\theta^T(t)} = \dfrac{g_\theta(t) . K(t,y)}{g_\theta(t) \int K(t,y)\,dy} = \dfrac{K(t,y)}{\int K(t,y')\,dy'}$ which is indp. of θ.

<u>Only if part:</u>

Let $p_\theta^{Y/t}(y)$ be independent of θ, say, $K(t,y)$.

Then, $p_\theta^{T,Y}(t,y) = K(t,y) . p_\theta^T(t)$

$\Rightarrow p_\theta(x) = p_\theta^{T,Y}(y(x), t(x)) . J\left(\frac{t_1, \ldots, t_r, y_1, \ldots, y_{n-r}}{x_1, x_2, \ldots, x_n}\right)$

$= p_\theta^T(t) . K(t(x), y(x)) . J\left(\frac{t_1, \ldots, t_r, y_1, \ldots, y_{n-r}}{x_1, x_2, \ldots, x_n}\right)$

$= g_\theta(t) . h(\underline{x})$, where $g_\theta(t) = p_\theta^T(t)$, $h(\underline{x}) = K(t(x), y(x)) . J\left(\frac{\cdots}{\cdots}\right)$.

Hence the theorem is proved.

Examples:

1. Suppose $x_1, x_2, \ldots, x_n$ are iid $N(\mu, \sigma^2)$, where μ, σ^2 unknown, $\theta = (\mu, \sigma^2)$

$$p_\theta(x) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2}\{\sum x_i^2 - 2\mu\sum x_i + n\mu^2\}}$$

$$= g_\theta\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2\right) \cdot h(\underline{x}), \text{ where } h(\underline{x}) = 1$$

$\Rightarrow T = (\sum x_i, \sum x_i^2)$ is sufficient for θ

$\Rightarrow T^* = (\bar{x}, \sum(x_i - \bar{x})^2)$ is also sufficient for θ, since T and T^* are in 1:1 relation.

2. Let $x_1, x_2, \ldots, x_n$ iid $N(\mu, \sigma^2)$.
   If σ^2 is known, $\bar{x}$ will be sufficient for μ.
   If μ is " , $\sum x_i^2$ (or $\sum(x_i - \bar{x})^2$) will be sufficient for σ^2.

3. Let $x_1, x_2, \ldots, x_n$ iid $R(\theta_1, \theta_2)$.

$$p_\theta(x) = \frac{1}{(\theta_2-\theta_1)^n} \quad \text{if } \theta_1 < x_{(1)} \le \cdots \le x_{(n)} < \theta_2, \ \theta_1 < \theta_2$$
$$= 0 \quad \text{otherwise.}$$

i.e. $p_\theta(x) = \frac{1}{(\theta_2-\theta_1)^n} u(x_{(1)} - \theta_1)\, u(\theta_2 - x_{(n)})$, where $u(x) = 0$ if $x < 0$
$= 1$ if $x > 0$.

Case I: θ_1 is known.

$p_\theta(x) = g_\theta(x_{(n)}) \cdot h(x)$, where $g_\theta(x_{(n)}) = \frac{1}{(\theta_2-\theta_1)^n} u(\theta_2 - x_{(n)})$,
$h(x) = u(x_{(1)} - \theta_1)$.

$\Rightarrow T = x_{(n)}$ is sufficient for θ_2.

Case II: θ_2 is known.

$p_\theta(x) = g_\theta(x_{(1)})\, h(x)$, where $g_\theta(x_{(1)}) = \frac{1}{(\theta_2-\theta_1)^n} u(x_{(1)} - \theta_1)$,
$h(x) = u(\theta_2 - x_{(n)})$

$\Rightarrow T = x_{(1)}$ is sufficient for θ_1.

Case-III: θ_1, θ_2 are both unknown.

$p_\theta(x) = g_\theta(x_{(1)}, x_{(n)}) \cdot h(x)$, where $h(x) = 1$

$\Rightarrow T = (x_{(1)}, x_{(n)})$ is sufficient for $\theta = (\theta_1, \theta_2)$.

Minimal Sufficient Statistics

Let x is a r.v. with pdf or pmf $p(x) \in \mathcal{P} = \{p_\theta(x) : \theta \in \Omega\}$. We want to make inference about unknown θ. For this we use a sufficient statistic $T = t(x)$ to x. We should further try to choose T such that it provides a more thorough reduction than any other sufficient statistic. Such a statistic T is called a minimal sufficient statistic.

Example: $\underset{\sim}{X} = (x_1, x_2, \cdots, x_n)$ be results of n Bernoullian trials, with success probability θ.

$$T_1 = (x_1, x_2, \cdots, x_n) = \underset{\sim}{x}$$
$$T_2 = (x_1 + x_2, x_3, \cdots, x_n)$$
$$T_3 = (x_1 + x_2 + x_3, x_4, \cdots, x_n)$$
$$- - - - - - - - - -$$
$$T_n = (x_1 + x_2 + \cdots + x_n)$$

By factorization theorem, all these statistics are sufficient for θ. But as T_n is a function of all other statistics T_i's, T_n gives the most thorough reduction of x. Hence T_n is a minimal sufficient statistic.

Defn: A sufficient statistic T is said to be minimal sufficient if it is a function of every other sufficient statistic, i.e. for any sufficient statistic T^* $\exists$ a function $S(\cdot) \ni t(x) = S(t^*(x))$ a.e.

If T be a minimal sufficient statistic and T^* be a one-to-one function of T, then T^* is also a minimal sufficient statistic.

Examples:

1. Let $\underset{\sim}{x} = (x_1, x_2, \cdots, x_n)$ be results of n Bernoullian trials with ~~failure~~ success probability θ.

$$p_\theta(x) = \theta^{\Sigma x_i} (1-\theta)^{n - \Sigma x_i}, \quad x_i = 0, 1.$$

Two points x, y with $p_\theta(y) > 0$ will belong to same coset of the minimal sufficient partition iff $\frac{p_\theta(y)}{p_\theta(x)}$ is independent of θ.

Now $\frac{p_\theta(y)}{p_\theta(x)} = \theta^{\Sigma y_i - \Sigma x_i} (1-\theta)^{\Sigma x_i - \Sigma y_i}$, which is independent of θ iff

$\Sigma x_i = \Sigma y_i$

$\Rightarrow \Sigma x_i$ is a minimal sufficient statistic.

2. $x_1, x_2, \cdots, x_n$ are iid $N(\theta_1, \theta_2)$, $\underset{\sim}{\theta} = (\theta_1, \theta_2)$.

$$p_\theta(\underset{\sim}{x}) = \text{Const.}\ e^{-\frac{1}{2\theta_2} \Sigma (x_i - \theta_1)^2}$$

Now $$\frac{p_\theta(y)}{p_\theta(x)} = e^{-\frac{1}{2\theta_2}\{\Sigma y_i^2 + n\theta_1^2 \cancel{-2\theta_1 \Sigma y_i} - \Sigma x_i^2 + 2\theta_1 \Sigma x_i - n\theta_1^2\}}$$
$$= e^{-\frac{1}{2\theta_2}[(\Sigma y_i^2 - \Sigma x_i^2) - 2\theta_1 n(\bar{y} - \bar{x})]}$$

This is independent of θ iff $\sum y_i^2 = \sum x_i^2$ & $\bar{y} = \bar{x}$.

$\Rightarrow T = (\bar{x}, \sum x_i^2)$ is a minimal sufficient statistic.

$\Rightarrow T^* = (\bar{x}, \sum (x_i - \bar{x})^2)$ is a minimal sufficient statistic.

Note: In examples 1 and 2, we find that dimension of θ is equal to dimension of minimal sufficient statistic. But this is not always true.

3. Suppose $(x_1, x_2, \dots, x_m) \sim N(\theta_1, \theta_2)$ and $(x_{m+1}, \dots, x_n) \sim N(\theta_1, \theta_3)$.

Here $\underset{\sim}{\theta} = (\theta_1, \theta_2, \theta_3)$.

Let $\underline{x} = (x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_n)$

$$p_\theta(\underline{x}) = \text{const. } e^{-\frac{1}{2\theta_2}\sum_{i=1}^{m}(x_i - \theta_1)^2 - \frac{1}{2\theta_3}\sum_{i=m+1}^{n}(x_i - \theta_1)^2}$$

$$\therefore \frac{p_\theta(\underline{y})}{p_\theta(\underline{x})} = \exp\Big[-\frac{1}{2\theta_2}\Big(\sum_{i=1}^{m} y_i^2 - \sum_{i=1}^{m} x_i^2\Big) + \frac{\theta_1}{\theta_2}\Big(\sum_{i=1}^{m} y_i - \sum_{i=1}^{m} x_i\Big) - \frac{1}{2\theta_3}\Big(\sum_{i=m+1}^{n} y_i^2 - \sum_{i=m+1}^{n} x_i^2\Big)$$
$$+ \frac{\theta_1}{\theta_3}\Big(\sum_{i=m+1}^{n} y_i - \sum_{i=m+1}^{n} x_i\Big)\Big]$$

This is independent of θ iff

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i, \quad \sum_{i=m+1}^{n} x_i = \sum_{i=m+1}^{n} y_i, \quad \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} y_i^2, \quad \sum_{i=m+1}^{n} x_i^2 = \sum_{i=m+1}^{n} y_i^2.$$

$\Rightarrow T = \Big(\sum_{i=1}^{m} x_i, \sum_{i=m+1}^{n} x_i, \sum_{i=1}^{m} x_i^2, \sum_{i=m+1}^{n} x_i^2\Big)$ is a minimal sufficient statistic.

Hence, dim. of $T = 4 > 3 =$ dim. of $\underline{\theta}$.

4. Let $\underset{\sim}{x} = (x_1, x_2, \dots, x_n) \sim N_n(\underset{\sim}{\mu}, \Sigma)$, where

$$\underset{\sim}{\mu} = \begin{pmatrix} n\theta_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \Sigma^{n\times n} = \begin{pmatrix} (n-1)+\theta_2^2 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (n-1)+\theta_2^2 & -\underline{\epsilon}' \\ -\underline{\epsilon} & I_{n-1} \end{pmatrix}$$

~~Show that $\bar{x}$ is a minimal sufficient statistic~~

$$p_\theta(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{n}{2}}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})}$$

$$\text{Here } \Sigma^{-1} = \frac{1}{\theta_2^2}\begin{pmatrix} 1 & \underline{\epsilon}' \\ \underline{\epsilon} & \theta_2^2 I_{n-1} + \underline{\epsilon}\,\underline{\epsilon}' \end{pmatrix}$$

$$= \frac{1}{\theta_2^2}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1+\theta_2^2 & 1 & \cdots & 1 \\ 1 & 1 & 1+\theta_2^2 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1+\theta_2^2 \end{pmatrix}$$

$$(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu}) = \frac{1}{\theta_2^2}\sum_{i=1}^{n}\sum \sigma^{ij}(x_i-\mu_i)(x_j-\mu_j),$$

$$= \frac{1}{\theta_2^2}\Big[(x_1-n\theta_1)^2\sigma^{11} + 2(x_1-n\theta_1)\sigma^{11}\sum_{i=2}^{n}x_i + \sigma^{11}\sum_{i=2}^{n}x_i^2 + \sum_{\substack{i,j=2\\(i\neq j)}}^{n}\sum x_i x_j \sigma_{ij}\Big]$$

$$= \frac{1}{\theta_2^2}\Big[\Big(x_1 - n\theta_1 + \sum_{i=2}^{n}x_i\Big)^2 + \theta_2^2\sum_{i=2}^{n}x_i^2\Big]$$

$$= \frac{1}{\theta_2^2}\Big[\Big(\sum_{i=1}^{n}x_i - n\theta_1\Big)^2 + \theta_2^2\sum_{i=2}^{n}x_i^2\Big]$$

$$= \frac{n^2}{\theta_2^2}(\bar{x}-\theta_1)^2 + \sum_{i=2}^{n}x_i^2$$

$$p_\theta(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\theta_2^n}e^{-\frac{1}{2}\left[\frac{n^2}{\theta_2^2}(\bar{x}-\theta_1)^2 + \sum_{i=2}^{n}x_i^2\right]}$$

$$\Rightarrow \frac{p_\theta(\underline{y})}{p_\theta(\underline{x})} = \exp\Big[-\frac{1}{2}\Big\{\frac{n^2}{\theta_2^2}\{\bar{y}^2-\bar{x}^2-2\theta_1(\bar{y}-\bar{x})\} + \sum_{i=2}^{n}y_i^2 - \sum_{i=2}^{n}x_i^2\Big]$$

which is independent of θ iff $\bar{y}=\bar{x}$.

$\Rightarrow T=\bar{x}$ is a minimal sufficient statistic. Here dimension of minimal sufficient statistic $=1 < 2 =$ dimension of $\underline{\theta}$.

Completeness

Consider the family of pmf or pdf $\mathcal{P} = \{p_\theta(\underline{x}) : \theta\in\Omega\}$. Then the family $\mathcal{P}$ is said to be complete if for any real independent function $f(x)$,

$$E_\theta[f(x)] = 0 \ \forall\theta \text{ ---- (1)}$$

$$\Leftrightarrow f(x) = 0 \text{ a.e. (regarding } \mathcal{P}) \text{ --- (2)}$$

i.e. $\nexists$ any non-zero function $f(x) \ni E_\theta\{f(x)\} = 0 \ \forall\theta$.

If (1) $\Rightarrow$ (2) only for bounded real valued functions $f(x)$, then $\mathcal{P}$ is said to be boundedly complete.

Note 1: Clearly $\mathcal{P}$ is complete $\Rightarrow$ $\mathcal{P}$ is boundedly complete.

But the converse is not necessarily true.

Example: X is discrete r.v. with $P_\theta[X=-1] = \theta$, $0<\theta<1$,

$P_\theta[X=x] = \theta^x(1-\theta)^2$, $x = 0,1,2,\dots,\infty$

$$0 = E_\theta[f(X)] = f(-1)\cdot\theta + \sum_{x=0}^{\infty}f(x)\theta^x(1-\theta)^2$$

$$\Rightarrow \sum_{x=0}^{\infty}f(x)\theta^x = -f(-1)\frac{\theta}{(1-\theta)^2} = -f(-1)\cdot\sum_{x=0}^{\infty}x\cdot\theta^x$$

$$\Leftrightarrow f(x) = -x f(-1),\ x = 0,1,2,\dots,\infty \text{ ----- } (*)$$

(by equating the co-efficients of θ^x from both sides)

If we define $f(-1) = c \neq 0$, then $f(x) = -cx$; $x = 0, 1, 2, \cdots$

Hence $E_\theta[f(X)] \neq 0$ with probability 1.

In this case, it is obvious that the function $f(x)$ is unbounded.

Hence for any unbounded function $f(x)$, $E_\theta f(X) \neq 0$

$\Rightarrow$ The family is not complete.

Now suppose we take $f(x)$ to be a bounded function.

Then, clearly, $f(-1) = 0$, since otherwise $f(x)$ becomes unbounded

$\Rightarrow f(x) = 0 \quad \forall x = 0, 1, 2, \cdots, \infty$.

$\Rightarrow$ The family is boundedly complete.

Let $T = t(x)$ be a statistic, and let

$\mathcal{P}^T = \{ p_\theta^T(t) : \theta \in \Omega \}$

= Induced family of probability distributions (Induced by the statistic).

Then T is said to be complete (boundedly complete) if $\mathcal{P}^T$ is complete (boundedly complete).

i.e. $E_\theta f(T) = 0 \quad \forall \theta$

$\Rightarrow f(t) = 0$ a.e. (regarding $\mathcal{P}^T$)

[$f(T)$ being necessarily bounded for bounded completeness].

<u>Note2</u>: Let T and T^* be two statistics such that T^* is a function of T.

Then T is complete $\Rightarrow T^*$ is complete

<u>Proof</u>: Let $T^* = h(T)$

Now, $E_\theta[f(T^*)] = 0 \quad \forall \theta$

$\Leftrightarrow E_\theta\{f(h(T))\} = 0 \quad \forall \theta$

$\Leftrightarrow E_\theta[g(T)] = 0 \quad \forall \theta$, where $g(T) = f\{h(T)\}$

$\Rightarrow g(t) = 0$ a.e. [$\because T$ is complete]

i.e. $f(h(t)) = 0$ a.e.

i.e. $f(t^*) = 0$ a.e.

$\Rightarrow T^*$ is complete.

<u>Note 3.</u> If T and T^* be equivalent statistics, ~~then~~ and T is complete $\Leftrightarrow$ T^* is complete.

<u>Proof:</u> This follows from Note 2 and the fact that T is a function of T^* and vice-versa.

<u>Note 4:</u> Let $\mathcal{P}_0 = \{p_\theta(x) : \theta \in \Omega_0\}$ and $\mathcal{P} = \{p_\theta(x) : \theta \in \Omega\}$, $\Omega_0 \subset \Omega$.

Then, $\mathcal{P}_0$ is complete $\Rightarrow$ $\mathcal{P}$ is complete.

i.e. $\nexists$ any set $S \ni P_\theta[X \in S] = 0 \ \forall \theta \in \Omega$, but $P_\theta[X \in S] > 0$ for some $\theta \in \Omega - \Omega_0$.

<u>Proof:</u> $E_\theta[f(x)] = 0 \quad \forall \theta \in \Omega$

$\Rightarrow E_\theta[f(x)] = 0 \quad \forall \theta \in \Omega_0$

$\Rightarrow f(x) = 0$ a.e. (regarding $\mathcal{P}_0$)

$\Leftrightarrow f(x) = 0$ a.e. (regarding $\mathcal{P}$)

$\Rightarrow \mathcal{P}$ is complete.

<u>Note 5:</u> Completeness of $\mathcal{P}$ does not necessarily imply the completeness of $\mathcal{P}_0$.

<u>Example:</u> Let $(x_1, x_2, \ldots, x_m) \sim N(\theta_1, \theta_2)$ and $(x_{m+1}, \ldots, x_{m+n}) \sim N(\theta_3, \theta_4)$.

Let $T = (\bar{x}_1, \bar{x}_2, s_1^2, s_2^2)$, where $\bar{x}_1 = \frac{1}{m}\sum_{i=1}^{m} x_i$, $\bar{x}_2 = \frac{1}{n}\sum_{j=1}^{n} x_{m+j}$,

$$s_1^2 = \sum_{i=1}^{m}(x_i - \bar{x}_1)^2, \quad s_2^2 = \sum_{j=1}^{n}(x_{m+j} - \bar{x}_2)^2$$

T is complete (to be shown later) when $\Omega = \{(\theta_1, \theta_2, \theta_3, \theta_4), -\infty < \theta_1, \theta_3 < \infty, \theta_2, \theta_4 > 0\}$.

We consider $\Omega_0 \subset \Omega$, where $\Omega_0 = \{(\theta_1, \theta_2, \theta_3, \theta_4), -\infty < \theta_1 = \theta_3 < \infty, \theta_2, \theta_4 > 0\}$.

Then T is not complete for $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Omega_0$.

Since if we consider the function $f(T) = \bar{x}_1 - \bar{x}_2$

$E_\theta[f(T)] = E_\theta[\bar{x}_1] - E_\theta(\bar{x}_2) = 0 \quad \forall \theta \in \Omega_0$

$\not\Rightarrow f(t) = 0$ a.e.

i.e. $\bar{x}_1 \neq \bar{x}_2$ a.e.

Examples of complete family

1. Binomial family.

$$p_\theta(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \quad 0<\theta<1,\ x = 0, 1, \cdots, n.$$

$$0 = E_\theta[f(X)] = \sum_{x=0}^{n} f(x) \cdot \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad \forall \theta \in (0,1)$$

$$\Rightarrow \sum_{x=0}^{n} a(x) \lambda^x = 0 \quad \forall \lambda \in (0,\infty)$$

where $\cancel{a(x) =}\ a(x) = f(x)\binom{n}{x}\cdot$

and $\lambda = \frac{\theta}{1-\theta}$.

$\Rightarrow a(x) = 0 \quad \forall x = 0, 1, \cdots, n$, since $0 < \lambda^x < \infty$.

$\Leftrightarrow f(x)\binom{n}{x} = 0 \quad \forall x = 0, 1, \cdots, n$

$\Leftrightarrow f(x) = 0 \quad \forall x = 0, 1, \cdots, n$, since $\binom{n}{x} > 0$.

$\Rightarrow$ The Bionomial family is complete.

<u>Application:</u> $X = (X_1, X_2, \cdots, X_n) \rightarrow$ results of n independent Bernoullian trials with success θ, $0<\theta<1$.

$\Rightarrow T = \sum_{i=1}^{n} X_i \sim Bin(n, \theta)$

$\Rightarrow T$ is complete.

2. Poisson family

$$p_\theta(x) = e^{-\theta} \frac{\theta^x}{x!}, \quad x = 0, 1, 2, \cdots, \infty, \ \theta \in (0, \infty)$$

Then $0 = E_\theta[f(X)] = \sum_{x=0}^{\infty} f(x) \cdot e^{-\theta} \cdot \frac{\theta^x}{x!}, \quad \theta \in (0,\infty)$

$$\Rightarrow \sum_{x=0}^{\infty} \theta^x \frac{f(x)}{x!} = 0, \forall \theta \in (0,\infty) \ [\text{since } e^{-\theta} > 0].$$

$\Rightarrow \frac{f(x)}{x!} = 0 \quad \forall x = 0, 1, \cdots, \infty$, since $\theta^x > 0$.

$\Rightarrow f(x) = 0 \quad \forall x = 0, 1, \cdots, \infty$, since $x! > 0$.

$\Rightarrow$ Poisson family is complete.

<u>Application:</u> If $X_1, X_2, \cdots, X_n$ are iid $\sim$ Poisson(θ), Then $T = \sum_{i=1}^{n} X_i \sim$ Poisson$(n\theta)$

$\Rightarrow T$ is complete.

3. Hypergeometric family

$$p_\theta(x) = \frac{\binom{\theta}{x}\binom{N-\theta}{n-x}}{\binom{N}{n}}, \quad x = 0,1,\dots,\min(n,\theta), \quad \theta = 0,1,2,\dots,N.$$

$$0 = E_\theta[f(x)] = \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} f(x)\cdot\binom{\theta}{x}\binom{N-\theta}{n-x}, \quad \theta = 0,1,2,\dots,N.$$

$$\Rightarrow \sum_{x=0}^{n} f(x)\binom{\theta}{x}\binom{N-\theta}{n-x} = 0, \quad \theta = 0,1,2,\dots,N. \quad \dots\dots (1)$$

For $\theta = 0$, (1) $\Rightarrow \binom{N}{n} f(0) = 0 \Rightarrow f(0) = 0$, since $\binom{N}{n} > 0$

For $\theta = 1$, (1) $\Rightarrow \binom{N-1}{n} f(0) + \binom{N-1}{n-1} f(1) = 0 \Rightarrow f(1) = 0$.

For $\theta = 2$, (1) $\Rightarrow f(2) = 0$

— — — — — — — —

For $\theta = n$, (1) $\Rightarrow f(n) = 0$.

i.e. $f(x) = 0 \quad \forall\, x = 0,1,2,\dots,n$.

$\Rightarrow$ The family is complete.

Application:

Suppose, we have N objects of which θ are defective.

We draw n objects by SRSWOR.

Let $x_i = 1$ if the i^{th} object is defective
$\quad\;\; = 0$ " " " " non-defective.

$T = \sum_{i=1}^{n} x_i \sim$ Hypergeometric (N, n, θ).

$\Rightarrow T$ is a complete statistic.

Example $x_1, x_2, \dots, x_n$ iid $\sim R(\theta_1, \theta_2)$; $-\infty < \theta_1 < \theta_2 < \infty$

Let $T_1 = x_{(1)}$, $T_2 = x_{(n)}$, $T = (T_1, T_2)$.

Case-I: θ_1 is known but θ_2 is unknown.

Let $\theta = \theta_2$.

$$p_\theta^{T_2}(t_2) = \frac{n}{(\theta-\theta_1)^n}(t_2-\theta_1)^{n-1}; \quad \theta_1 < t_2 < \theta.$$

$$0 = E_\theta[f(T_2)] = \frac{n}{(\theta-\theta_1)^n}\int_{\theta_1}^{\theta} f(t_2)(t_2-\theta_1)^{n-1}\,dt_2, \quad \forall\,\theta \in (\theta_1, \infty).$$

$$\Rightarrow \int_{\theta_1}^{\theta} f(t_2)(t_2-\theta_1)^{n-1}\,dt_2 = 0 \quad \forall\,\theta \in (\theta_1, \infty)$$

$$\Rightarrow f(\theta)(\theta-\theta_1)^{n-1} = 0 \quad \forall\,\theta \in (\theta_1, \infty) \quad [\text{Diff. w.r.t. } \theta].$$

$$\Rightarrow f(\theta) = 0 \quad \forall\,\theta \in (\theta_1, \infty)$$

$$\Rightarrow f(t_2) = 0 \quad \forall\, t_2 \in (\theta_1, \theta), \; \theta \in (\theta_1, \infty).$$

T_2 is a complete statistic.

Case-II: θ_2 is known, but $\theta=\theta_1$ is unknown.

$$p_\theta^{T_1}(t_1) = \frac{n}{(\theta_2-\theta)^n}(\theta_2-t_1)^{n-1}; \quad \theta<t_1<\theta_2$$

Now $0 = E_\theta[f(T_1)] = \int_\theta^{\theta_2} f(t_1)\cdot\frac{n}{(\theta_2-\theta)^n}(\theta_2-t_1)^{n-1}dt_1, \ \forall \theta\in(-\infty,\theta_2)$

$\Rightarrow \int_\theta^{\theta_2} f(t_1)\cdot(\theta_2-t_1)^{n-1}dt_1=0, \ \forall\theta\in(-\infty,\theta_2)$

$\Rightarrow -f(\theta)(\theta_2-\theta)^{n-1}=0 \quad \forall\theta\in(-\infty,\theta_2)$ [Diff. w.r.t. θ]

$\Rightarrow f(\theta)=0 \quad \forall\theta\in(-\infty,\theta_2)$

$\Rightarrow f(T_1)=0 \quad \forall T_1\in(\theta,\theta_2),\ \theta\in(-\infty,\theta_2)$

$\therefore T_1$ is a complete statistic.

Case-III: $\theta=(\theta_1,\theta_2)$ is unknown

Here $p_\theta^T(t) = \frac{n(n-1)}{(\theta_2-\theta_1)^n}(t_2-t_1)^{n-2}; \quad \theta_1<t_1<t_2<\theta_2$

Now $0 = E_\theta[f(T)] = \frac{n(n-1)}{(\theta_2-\theta_1)^n}\int_{\theta_1}^{\theta_2}\int_{t_1}^{\theta_2} f(t_1,t_2)(t_2-t_1)^{n-2}dt_2dt_1,$

$\forall -\infty<\theta_1<\theta_2<\infty.$

$\Rightarrow \int_{\theta_1}^{\theta_2} h(t_1,\theta_2)dt_1=0, \quad \forall -\infty<\theta_1<\theta_2<\infty$ [where $h(t_1,\theta_2)=\int_{t_1}^{\theta_2} f(t_1,t_2)(t_2-t_1)^{n-2}dt_2$].

i.e. $h(\theta_1,\theta_2)=0 \quad \forall -\infty<\theta_1<\theta_2<\infty.$
[Diff. w.r.t. θ_1]

or, $\int_{\theta_1}^{\theta_2} f(\theta_1,t_2)(t_2-\theta_1)^{n-2}dt_2=0 \quad \forall -\infty<\theta_1<\theta_2<\infty$

$\Rightarrow f(\theta_1,\theta_2)(\theta_2-\theta_1)^{n-2}=0 \quad \forall -\infty<\theta_1<\theta_2<\infty$ [Diff. w.r.t. θ_2].

$\Rightarrow f(\theta_1,\theta_2)=0 \quad \forall -\infty<\theta_1<\theta_2<\infty.$

or, $f(t_1,t_2)=0 \quad \forall \theta_1<t_1<t_2<\theta_2,\ -\infty<\theta_1<\theta_2<\infty$

$\Rightarrow T=(T_1,T_2)$ is complete.

Some Integral Transforms

Let $f(x)$ be a continuous function of $x \in (0, \infty)$.

Let $\Phi(t) = \int_0^\infty e^{-tx} f(x)\, dx$.

This is called Unilateral Laplace Transformation of $f(x)$.

Let $\Phi(t) = \int_{-\infty}^{\infty} e^{-tx} f(x)\, dx$ [when $x \in (-\infty, \infty)$] is called Bilateral Laplace Transformation of $f(x)$.

$\Phi(t) = \int_0^\infty x^{t-1} f(x)\, dx \longrightarrow$ Mellin's Transform of $f(x)$.

$\Phi(t) = \int_0^\infty \frac{1}{t+x} f(x)\, dx \longrightarrow$ Stiltjes Transform of $f(x)$.

The integral transform of zero is zero.

A common Uniqueness property of these integral transforms:

If $\Phi_1(t)$ and $\Phi_2(t)$ be integral transforms of $f_1(x)$ and $f_2(x)$ respectively, then

$\Phi_1(t) = \Phi_2(t) \Rightarrow f_1(x) = f_2(x)$ a.e.

Corollary: If the integral transform of a function $f(x)$ is zero, then $f(x) = 0$ a.e.

Example of complete families

1. $N(\theta,1)$ family.

$$p_\theta(x) = \text{const. } e^{-\frac{1}{2}(x-\theta)^2} \; ; \; -\infty<\theta<\infty.$$

$$0 = E_\theta[f(X)]$$
$$= \text{const. } \int_{-\infty}^{\infty} f(x)\cdot e^{-\frac{x^2}{2}+\theta x}\, dx \;, \; \forall \theta \in (-\infty,\infty)$$

$$\Rightarrow \int_{-\infty}^{\infty} \underbrace{\{ f(x)\, e^{-\frac{x^2}{2}} \} e^{\theta x}\, dx}_{\text{Bilateral Laplace Transformation}} = 0 \;, \; \forall \theta \in (-\infty,\infty)$$

$$\Rightarrow f(x)\cdot e^{-\frac{x^2}{2}} = 0 \quad a.e.$$

i.e. $f(x) = 0$ a.e.

$\Rightarrow$ The family is complete.

Application

$X_1, X_2, \ldots, X_n$ iid $\sim N(\theta,1)$

Consider $T = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$\sqrt{n}\,\bar{X} \sim N(\sqrt{n}\,\theta, 1)$

$\Rightarrow \sqrt{n}\,\bar{X}$ is a complete statistic

$\Rightarrow \bar{X}$ " " " " .

2. $N(0,\theta)$ family.

$$p_\theta(x) = \text{const. } e^{-\frac{x^2}{2\theta}} \; ; \; 0<\theta<\infty$$

For This family

$E_\theta[f(X)] = 0$ for any odd functions like $x, x^3, x^5, \ldots$

$\Rightarrow$ The family will not bee complete, Since $p_\theta(x)$ is an even function.

Suppose, we consider $T = X^2$.

$$p_\theta^T(t) = (2\pi\theta t)^{-\frac{1}{2}}\, e^{-\frac{t}{2\theta}}$$

$$0 = E_\theta f(T) \quad \forall \theta \in (0,\infty)$$

$$\Leftrightarrow \int_0^{\infty} f(t)\, e^{-\frac{t}{2\theta}}\, t^{-\frac{1}{2}}\, dt = 0 \quad \forall \theta \in (0,\infty).$$

$\Rightarrow f(t)\, t^{-\frac{1}{2}} = 0$ a.e. [by Unilateral Laplace]

$\Rightarrow f(t) = 0$ a.e.

i.e. T is a complete statistic.

Application: $X_1, X_2, \ldots, X_n$ be iid $\sim N(0,\theta)$, Then $T = \sum X_i^2$ is complete.

3. $p_\theta(x) = \frac{1}{2^\theta \Gamma(\theta)} e^{-\frac{x}{2}} x^{\theta-1}$; $\theta \in (0,\infty)$

Now $0 = E_\theta f(x) = \text{Const.} \int_0^\infty \{f(x) e^{-\frac{x}{2}}\} x^{\theta-1} dx$

$\Rightarrow f(x) \cdot e^{-\frac{x}{2}} = 0$ a.e. [by Mellin's transformation]

$\Rightarrow f(x) = 0$ a.e.

$\Rightarrow$ The family is complete.

Complete Sufficiency

A statistic T is said to be complete sufficient for $\{p_\theta(x) : \theta \in \Omega\}$ if

i) T is sufficient for θ

ii) T is a complete statistic.

Note 1: All sufficient statistics are not complete.

Example: $N(\theta_1, \theta_2) \downarrow x_1, x_2, \ldots, x_m$; $\bar{x}_1, s_1^2$ $\qquad$ $N(\theta_1, \theta_3) \downarrow x_{m+1}, \ldots, x_n$; $\bar{x}_2, s_2^2$

$T = (\bar{x}_1, \bar{x}_2, s_1^2, s_2^2)$ is a minimal sufficient statistic for $\theta = (\theta_1, \theta_2, \theta_3)$.

But T is not complete, since for $f(T) = \bar{x}_1 - \bar{x}_2$,

$E_\theta f(T) = 0 \quad \forall\ \theta$

$\not\Rightarrow \bar{x}_1 = \bar{x}_2$ a.e.

$\Rightarrow T$ is not complete.

Note 2: If a sufficient statistic T is complete, it is minimal sufficient.

Proof: Let T^* be any minimal sufficient statistic. We shall show that T is equivalent to T^*.

Since T^* is a minimal sufficient statistic, T^* will be a function of any other sufficient statistic, and hence a function of T.

Let $\phi(T) = T - E(T/T^*)$

T^* is sufficient $\Rightarrow E(T/T^*)$ is independent of θ.

$\Rightarrow \phi(T)$ is a function of T only.

Also, $E_\theta \phi(T) = E_\theta(T) - E_\theta E(T/T^*) = E_\theta(T) - E_\theta(T) = 0 \quad \forall\ \theta$

$\Rightarrow \phi(T) = 0$ a.e. (since T is complete)

$\Rightarrow T = E(T/T^*)$ a.e.

i.e. T is a function of T^* a.e.
Hence T and T^* are equivalent statistics
$\Rightarrow T$ is minimal sufficient.

Exponential family of Distributions

Case-I: The case of a single parameter

A family $\mathcal{P} = \{ p_\theta(x) : \theta \in \Omega \}$ is said to be a one-parameter exponential family if

$$p_\theta(x) = K(\theta)\, e^{Q(\theta)\cdot t(x)} \cdot h(x) \quad \text{------ (1)}$$

Ω = an open subset of $\mathbb{R}$.

where $K(\theta)$, $Q(\theta)$ are real valued functions of θ, $t(x)$, $h(x)$ are real valued functions of x.

Examples:

1. $X = (x_1, x_2, \cdots, x_n) \to$ results of n independent Bernoullian trials with probability of success θ, $\theta \in (0,1)$, $x_i = 0$ or 1, $i = 1(1)n$.

$$\begin{aligned} p_\theta(x) &= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \\ &= (1-\theta)^n e^{(\sum x_i)\ln\frac{\theta}{1-\theta}} \\ &= K(\theta)\, e^{Q(\theta).t(x)} h(x), \text{ where } Q(\theta) = \ln\frac{\theta}{1-\theta},\ t(x) = \sum x_i,\ h(x) = 1 \end{aligned}$$

$\to$ One-parameter exponential family.

2. $x_1, x_2, \cdots, x_n$ iid Poisson (θ).

$$p_\theta(x) = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!} = K(\theta).\, e^{Q(\theta)\,t(x)}.\, h(x), \text{ where } K(\theta) = e^{-n\theta},\ Q(\theta) = \ln\theta,\ t(x) = \sum x_i,\ h(x) = \frac{1}{\prod_{i=1}^{n} x_i!}$$

$\to$ One-parameter exponential family.

3. $x_1, x_2, \cdots, x_n$ iid $\sim N(\theta, 1)$.

$$p_\theta(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2} = e^{-\frac{n\theta^2}{2}}.\, e^{\theta\sum x_i}\, \frac{1}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{\sum x_i^2}{2}} = K(\theta).\, e^{Q(\theta).t(x)}.\, h(x)$$

$\to$ One-parameter exponential family.

4. $x_1, x_2, \cdots, x_n$ iid $\sim N(0, \theta)$

$$p_\theta(x) = \frac{1}{(2\pi\theta)^{\frac{n}{2}}}\, e^{-\frac{1}{2\theta}\sum x_i^2} = K(\theta)\, e^{Q(\theta).t(x)}.\, h(x);\ \text{where } Q(\theta) = \frac{1}{2\theta},\ t(x) = \sum x_i^2,\ h(x) = 1.$$

$\to$ One-parameter exponential family.

Result 1: If $p_\theta(x)$ is of the form (1), then $T = t(X)$ is a complete sufficient statistic.

[Therefore, In examples (1), (2) and (3), $T = \Sigma x_i$ and in example (4), $T = \Sigma x_i^2$ is a complete sufficient statistic.]

Proof: The sufficiency of T follows from factorization Theorem.

To prove completeness, we first note that

$$p_\theta^T(t) = K(\theta) \cdot e^{Q(\theta).t} H(t)$$

[Proof: Discrete case

$$p_\theta(t) = \sum_{x:\, t(x)=t} p_\theta(x) = K(\theta)\, e^{Q(\theta).t} \sum_{x:\, t(x)=t} h(x) = K(\theta)\, e^{Q(\theta).t} H(t)$$

Absolutely continuous case

Let $X = (X_1, X_2, \cdots, X_n)$, and let there exist $Y_1, Y_2, \cdots, Y_{n-1}$ ∋ the transformation $X \longrightarrow (T, Y_1, Y_2, \cdots, Y_{n-1})$ is 1:1.

Then, $p_\theta^{T, Y_1, Y_2, \cdots, Y_{n-1}}(t, y_1, \cdots, y_{n-1})$

$$= p_\theta\left(x_1(t, y_1, \cdots, y_{n-1}), x_2(t, y_1, \cdots, y_{n-1}), \cdots, x_n(t, y_1, \cdots, y_{n-1})\right) \times J\left(\frac{x_1, x_2, \cdots, x_n}{t, y_1, \cdots, y_{n-1}}\right)$$

$$= K(\theta) \cdot e^{Q(\theta).t} \cdot h\left(x_1(t, y_1, \cdots, y_{n-1}), \cdots, x_n(t, y_1, \cdots, y_{n-1})\right) \cdot J.$$

$$\therefore p_\theta^T(t) = \int p_\theta^{T, Y_1, \cdots, Y_{n-1}}(t, y_1, \cdots, y_{n-1})\, dy_1\, dy_2 \cdots dy_{n-1}.$$

$$= K(\theta)\, e^{Q(\theta).t} \int h\left(x_1(t, y_1, \cdots, y_{n-1}), \cdots x_n(t, y_1, \cdots, y_{n-1})\right). J .\, dy_1 dy_2 \cdots dy_{n-1}.$$

$$= K(\theta).\, e^{Q(\theta).t} H(t), \text{ say}]$$

Then, $0 = E_\theta[f(T)] = K(\theta) \int f(t)\, e^{Q(\theta).t} H(t)\, dt$

$\Rightarrow \int f(t).H(t)\, e^{Q(\theta).t}\, dt = 0$

$\Rightarrow f(t)\, H(t) = 0$ a.e.

$\Rightarrow f(t) = 0$ a.e., since $H(t) > 0$.

$\Rightarrow$ T is a complete statistic.

Case-II: Case of multi-parameter exponential family.

A family $\mathcal{P} = \{p_\theta(x) : \theta \in \Omega\}$, $\theta = (\theta_1, \theta_2, \cdots, \theta_k)$, is said to be a multiparameter exponential family if

$$p_\theta(x) = K(\theta) \cdot e^{\underline{Q}(\theta)' \underline{t}(x)} \cdot h(x), \quad \Omega = \text{an open subset of } \mathbb{R}^k,$$

where $K(\theta)$ and components of $Q(\theta) = (Q_1(\theta), Q_2(\theta), \cdots, Q_k(\theta))'$ are real valued functions of θ, $h(x)$ and components of $t(x) = (t_1(x), \cdots, t_k(x))'$ are real valued functions of x.

Examples:

1. $x_1, x_2, \cdots, x_n$ iid $\sim N(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$.

$$p_\theta(x) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2}\sum_i (x_i-\mu)^2}$$

$$= \frac{e^{-\frac{n\mu^2}{2\sigma^2}}}{\sigma^n} e^{-\frac{\sum x_i^2}{2\sigma^2} + \frac{\mu}{\sigma^2}\cdot \sum x_i} \cdot \frac{1}{(2\pi)^{\frac{n}{2}}}$$

$$= K(\theta)\, e^{\underset{\sim}{Q}(\theta)' \underset{\sim}{t}(x)} h(x), \text{ where } \underset{\sim}{Q}(\theta) = \left(\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}\right)' \text{ and } \underset{\sim}{t}(x) = \left(\sum x_i\ \sum x_i^2\right).$$

$\Rightarrow$ 2-parameter exponential family.

2. $N(\theta_1, \theta_2) \downarrow (x_1, x_2, \cdots, x_m)$ $\quad$ $N(\theta_3, \theta_4) \downarrow (x_{m+1}, \cdots, x_{m+n})$

$$p_\theta(x) = \frac{1}{(2\pi\theta_2)^{\frac{m}{2}}} e^{-\frac{1}{2\theta_2}\sum_{i=1}^m (x_i-\theta_1)^2} \cdot \frac{1}{(2\pi\theta_4)^{\frac{n}{2}}} e^{-\frac{1}{2\theta_4}\sum_{i=1}^n (x_{m+i}-\theta_3)^2}$$

$$= \frac{e^{-\frac{m\theta_1^2}{2\theta_2} - \frac{n\theta_3^2}{2\theta_4}}}{\theta_2^{\frac{m}{2}}\ \theta_4^{\frac{n}{2}}} \cdot e^{\left(\underset{Q_1(\theta)}{\frac{\theta_1}{\theta_2}} \underset{t_1(x)}{\sum x_i} + \underset{Q_2(\theta)}{\frac{\theta_3}{\theta_4}} \underset{t_2(x)}{\sum x_{m+i}} - \frac{\sum x_i^2}{2\theta_2} - \frac{\sum x_{m+i}^2}{2\theta_4}\right)}$$

$$= K(\theta)\, e^{\underset{\sim}{Q}(\theta)' \underset{\sim}{t}(x)}\, h(x)$$

$\rightarrow$ 4-parameter exponential family.

3. $x_1, x_2, \cdots, x_n$ iid $\sim N_p(\underset{\sim}{\mu}, \Sigma)$.

$\theta = (\underset{\sim}{\mu}, \Sigma)$, $\Sigma = ((\sigma_{ij}))$, $\sigma_{ij} = \sigma_{ji}$

$\therefore$ Σ contains $p + \frac{p(p-1)}{2} = \frac{p(p+1)}{2}$ distinct elements.

$\therefore$ θ contains $p + \frac{p(p+1)}{2} = \frac{p(p+3)}{2}$ distinct elements.

$\Omega = \{\theta : -\infty < \mu_i < \infty, i = 1(1)p,\ 0 < \sigma_i < \infty, i = 1(1)p, -\infty < \sigma_{ij} < \infty, 1 \le i < j \le p\}$

$$p_\theta(x) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{n}{2}}} e^{-\frac{1}{2}\sum_{\alpha=1}^n (\underset{\sim}{x}_\alpha - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x}_\alpha - \underset{\sim}{\mu})}$$

Let $D = \sum_{\alpha=1}^n (\underset{\sim}{x}_\alpha - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x}_\alpha - \underset{\sim}{\mu}) = \sum_{\alpha=1}^n \underset{\sim}{x}_\alpha' \Sigma^{-1} \underset{\sim}{x}_\alpha - 2\underset{\sim}{\mu}' \Sigma^{-1} \sum_{\alpha=1}^n \underset{\sim}{x}_\alpha + n \underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\mu}$

Let $a_{ij} = \sum_{\alpha=1}^n (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)$, where $\underset{\sim}{\bar{x}} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{pmatrix}$, $\underset{\sim}{x}_\alpha = \begin{pmatrix} x_{1\alpha} \\ \vdots \\ x_{p\alpha} \end{pmatrix}$

Let further $\Sigma^{-1} = ((\sigma^{ij}))$

Then $D = \sum_i \sum_j \sigma^{ij} \left(\sum_\alpha x_{i\alpha} x_{j\alpha}\right) + n \underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\mu} - 2n \underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\bar{x}}$ (check)

$$\therefore p_\theta(x) = \frac{1}{(2\pi)^{\frac{np}{2}} |\Sigma|^{\frac{n}{2}}} e^{-\frac{n}{2}\underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\mu}}\ e^{\underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\bar{x}} - \frac{1}{2}\sum_i \sum_j \sigma^{ij} \left(\sum_\alpha x_{i\alpha} x_{j\alpha}\right)}$$

$$= K(\theta) \cdot e^{\sum_{j=1}^k Q_j(\theta) t_j(x)} \cdot h(x);\quad k = \frac{p(p+3)}{2},\ (t_1(x), t_2(x), \cdots, t_k(x)) = \left(\underset{\sim}{\bar{x}}, \sum_{\alpha=1}^n x_{i\alpha} x_{j\alpha}\right),\ 1 \le i \le j \le p.$$

$\rightarrow$ K-parameter exponential family.

Result 2 (Connected to one parameter family)

Let $x_1, x_2, \ldots, x_n$ be iid with common pdf $f_\theta(x)$. (θ is unidimensional), and let a sufficient statistic T (of dimension 1) exist for the family $\{p_\theta(x) : \theta \in \Omega\}$, where $p_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i)$. Then if the range of x_i is independent of θ, under certain regularity condition $f_\theta(x)$ and hence $p_\theta(x)$ must be of the exponential form.

Proof: Since T is sufficient, by factorization Theorem we can write

$$p_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i) = g_\theta(t) \cdot h(x)$$

$$\text{or, } \ln p_\theta(x) = \sum \ln f_\theta(x_i) = \ln g_\theta(t) + \ln h(x) \quad \cdots\cdots (1)$$

Regularity condition assummed: g and f are differentiable w.r.t. θ and x_i's.

Differentiating (1) w.r.t. θ we get

$$\sum \frac{\partial}{\partial \theta} \ln f_\theta(x_i) = \frac{\partial}{\partial \theta} \ln g_\theta(t) = K_\theta(t), \text{ say} \quad \cdots (2)$$

Now (2) is true for all θ and hence true for any particular θ, say $\theta = \theta_0$. So we get

$$\sum \frac{\partial}{\partial \theta} \ln f_\theta(x_i) \Big|_{\theta=\theta_0} = K_\theta(t) \Big|_{\theta=\theta_0}$$

$$\text{or, } \sum u(x_i) = K(t) \quad \cdots\cdots (3)$$

Since by putting a particular value of θ, $\frac{\partial}{\partial \theta} \ln f_\theta(x_i)\big|_{\theta=\theta_0} = u(x_i)$ is independent of θ and similarly $K_\theta(t)\big|_{\theta=\theta_0} = K(t)$ is independent of θ.

Differentiating (3) w.r.t. x_i we get

$$\frac{du(x_i)}{dx_i} = \frac{dK(t)}{dt} \cdot \frac{dt}{dx_i} \quad \cdots\cdot (4)$$

Differentiating (2) w.r.t. x_i, we get,

$$\frac{\partial^2 \ln f_\theta(x_i)}{\partial \theta \partial x_i} = \frac{\partial K_\theta(t)}{\partial t} \cdot \frac{\partial t}{\partial x_i} \quad \cdots\cdot (5)$$

Dividing (5) by (4) we get

$$\frac{\dfrac{\partial^2 \ln f_\theta(x_i)}{\partial \theta \partial x_i}}{\dfrac{\partial u(x_i)}{\partial x_i}} = \frac{\dfrac{\partial K_\theta(t)}{\partial t} \cdot \cancel{\dfrac{\partial t}{\partial x_i}}}{\dfrac{\partial K(t)}{\partial t} \cdot \cancel{\dfrac{\partial t}{\partial x_i}}} \quad \cdots\cdots (6)$$

The R.H.S. of (6) is the same for all x_i, implying it is independent of x_i's and is a function of θ only.

Hence, (6) $\Rightarrow \dfrac{\partial^2 \ln f_\theta(x_i)}{\partial \theta \partial x_i} \Big/ \dfrac{\partial u(x_i)}{\partial x_i} = A(\theta)$, (say).

$$\text{i.e. } \frac{\partial^2 \ln f_\theta(x_i)}{\partial \theta \partial x_i} = A(\theta) \cdot \frac{du(x_i)}{dx_i} \quad \cdots\cdot (7)$$

Integrating (7) w.r.t. x_i

$$\frac{\partial \ln f_\theta(x_i)}{\partial \theta} = A(\theta) u(x_i) + B(\theta), \text{ where } B(\theta) = \text{constant of integration.}$$

Integrating the above w.r.t. θ, we get,

$$\ln f_\theta(x_i) = A^*(\theta) u(x_i) + B^*(\theta) + C^*(x_i), \text{ where } C^*(x_i) = \text{constant of integration}$$

$\Rightarrow f_\theta(x) = e^{A^*(\theta)\,u(x) + B^*(\theta) + C^*(x)}$

$= K(\theta)\, e^{Q(\theta)\,u(x)}\, h(x)$, where $e^{B^*(\theta)} = K(\theta)$, $A^*(\theta) = Q(\theta)$, $e^{C^*(x)} = h(x)$

Thus $f_\theta(x)$ is of the exponential form.

Also, $p_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i) = e^{nB^*(\theta) + A^*(\theta)\sum u(x_i) + \sum C^*(x_i)}$

$= K^*(\theta)\, e^{Q(\theta)\sum u(x_i)} \cdot e^{\sum C^*(x_i)}$

which is of the exponential form.

Results on Multiparameter exponential family

Result 1: If $p_\theta(x)$ is of the multiparameter exponential form, viz,

$p_\theta(x) = K(\theta)\, e^{\sum Q_i(\theta)\, t_i(x)}\, h(x)$,

then $T = (T_1, T_2, \cdots, T_k) = (t_1(x), t_2(x), \cdots, t_k(x))$ is a complete sufficient statistic.

Proof:- Sufficiency follows from factorization theorem.

To prove completeness, we first have to show that $p_\theta(t)$ is of the exponential form, viz,

$p_\theta^T(t) = K(\theta)\, e^{\sum Q_i(\theta) t_i}\, H(t)$

Proof of this is along the same line as in the single parameter case.

Completeness of T

Consider any function $f(T) \ni$

$E_\theta f(T) = 0 \quad \forall \theta$

$\Leftrightarrow \int f(t)\, e^{\sum Q_i(\theta) t_i}\, H(t)\, dt = 0$, where $dt = dt_1 dt_2 \cdots dt_k$

This integral is a Laplace Transform of $f(t)H(t)$.

$\Rightarrow f(t)H(t) = 0$ a.e. (by uniqueness property)

$\Rightarrow f(t) = 0$ a.e. since $H(t) > 0$ a.e.

$\Rightarrow T$ is complete //

By this ~~result~~ result, we see that

in example 1, $T = (\sum x_i, \sum x_i^2)$ is complete sufficient statistic and so is $T^* = (\bar{x}, s^2)$.

In example 2, $T = \left(\sum_{i=1}^{m} x_i, \sum_{i=m+1}^{m+n} x_i, \sum_{i=1}^{m} x_i^2, \sum_{i=m+1}^{m+n} x_i^2\right)$ is complete sufficient statistic, and so is $(\bar{x}_1, \bar{x}_2, s_1^2, s_2^2)$.

In example 3, $T' = \left(\sum_\alpha x_{i\alpha},\ i = 1(1)p,\ \sum_\alpha x_{i\alpha} x_{j\alpha},\ 1 \le i \le j \le p\right)$ is a complete sufficient statistic, and ~~so is $(\bar{x}_1, \bar{x}_2, s_1^2, s_2^2)$~~ so is $T^* = (\bar{x}_1, \cdots, \bar{x}_p, a_{ij};\ 1 \le i \le j \le p)$ and so is $(\bar{X}, A)$.

<u>Result 2:</u> If $x_1, x_2, \cdots, x_n$ be iid with common pdf $f_\theta(x)$ where range of x is independent of $\theta = (\theta_1, \theta_2, \cdots, \theta_k)$ and if a sufficient statistic T of dimension k $(\leq n)$ exists, then $f_\theta(x)$ must be of the multiparameter (k-parameters) exponential form under some regularity conditions.